Lattice gauge notes VII

Review of GW: start with Wilson-Dirac operator D_W. Select mass in supercritical region. Does this have anything to do with the parity breaking discussed a couple weeks ago? Then:

$$\gamma_5 D_W \gamma_5 = D_W^\dagger$$
$$V = D_W (D_W^\dagger D_W + \epsilon^2)^{-1/2}$$
$$V^\dagger V = 1$$
$$D = 1 + V$$
$$\gamma_5 D + D\gamma_5 - D\gamma_5 D = 0$$

Then the fermionic action $\overline{\psi} D \psi$ is invariant under

$$\psi \rightarrow e^{i\theta\gamma_5(1-D)}\psi$$
$$\overline{\psi} \rightarrow \overline{\psi} e^{i\theta\gamma_5}$$

Various points:
- measure not invariant
- several flavors, insert λ matrix: measure invariant if $\mathrm{Tr}\lambda = 0$
- exact zero eigenvalues of D define an index, $n = n_+ - n_-$
- this is a compact symmetry: $\theta = 2\pi$ does nothing
- freedom in where to put the D: kinematic symmetry?
- mass term $m\overline{\psi}(1 - D/2)\psi$ transforms nicely

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To get a feeling for these matrices and their eigenvalues, study a two by two matrix example with the GW structure. Take γ_5 of form

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Hermiticity condition is

$$D_W^\dagger = \gamma_5 D_W \gamma_5$$

A generic "Wilson" operator that satisfies this has four parameters. For a convenient parametrization consider

$$D_W = e^{ia_1\sigma_1/2 + ia_2\sigma_2/2}(a_0 + a_3\sigma_3)e^{ia_1\sigma_1/2 + ia_2\sigma_2/2} = U(a_0 + a_3\sigma_3)U$$

where I define the unitary matrix

$$U = e^{ia_1\sigma_1/2 + ia_2\sigma_2/2}$$

This makes
$$D_W^\dagger D_W = U^\dagger(a_0^2 + a_3^2 + 2a_0 a_3 \sigma_3)U$$

so the square root is easy to take, including the epsilon cutoff

$$(D_W^\dagger D_W)^{-1/2} = U^\dagger \begin{pmatrix} \frac{1}{\sqrt{(a_0+a_3)^2+\epsilon^2}} & 0 \\ 0 & \frac{1}{\sqrt{(a_0-a_3)^2+\epsilon^2}} \end{pmatrix} U$$

and thus

$$V = D_W (D_W^\dagger D_W)^{-1/2} = U \begin{pmatrix} \frac{a_0+a_3}{\sqrt{(a_0+a_3)^2+\epsilon^2}} & 0 \\ 0 & \frac{a_0-a_3}{\sqrt{(a_0-a_3)^2+\epsilon^2}} \end{pmatrix} U$$

Note that the γ_5 hermiticity is maintained, even when the cutoff ϵ is relevant.

When $a_0 + a_3$ and $a_0 - a_3$ are of the same sign, $V = U^2$ and $D = 1+U^2$. The eigenvectors are the eigenvectors of

$$a_1\sigma_1 + a_2\sigma_2 = \begin{pmatrix} 0 & a_1 - ia_2 \\ a_1 + ia_2 & 0 \end{pmatrix}$$

The eigenvalues of this are
$$\theta_\pm = \pm\sqrt{a_1^2 + a_2^2}$$

and the eigenvectors are
$$\psi_\pm = \begin{pmatrix} e^{-i\phi/2} \\ \pm e^{i\phi/2} \end{pmatrix}$$

with $\tan(\phi) = a_2/a_1$. As desired $\psi_\mp = \gamma_5 \psi_\pm$, and the eigenvalues of V are complex conjugate pairs, with $\lambda = e^{\pm i\theta} = e^{\pm i\sqrt{a_1^2+a_2^2}}$.

The other case corresponds to when $a_0 + a_3$ and $a_0 - a_3$ are of the opposite sign. Consider $a_0 + a_3 > 0$ and $a_0 - a_3 < 0$, so

$$V = U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U = \sigma_3$$

where I have used $U\sigma_3 = \sigma_3 U^\dagger$. Thus the eigenvalues are ± 1. This is the "instanton" case.

To study the transition between the cases, let a_3 pass through $a_0 > 0$. In particular, let $-1 \le x \le 1$ and consider

$$V = U \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} U = U^2(1+x)/2 + (1-x)\sigma_3/2$$

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To proceed to a specific example, try

$$U^2 = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

giving

$$V = \begin{pmatrix} (1 + c - x + cx)/2 & -s(1+x)/2 \\ s(1+x)/2 & (-1 + c + x + cx)/2 \end{pmatrix}$$

Look at

$$\begin{aligned}
|V - \lambda| &= ((1 + c - x + cx)/2 - \lambda)((-1 + c + x + cx)/2 - \lambda) + s^2(1+x)^2/4 \\
&= \lambda^2 - \lambda(c + cx) + (c + cx)^2/4 - (1-x)^2/4 + s^2(1+x)^2/4 \\
&= \lambda^2 - c(1+x)\lambda + (1+x)^2/4 - (1-x)^2/4 \\
&= \lambda^2 - c(1+x)\lambda + x
\end{aligned}$$

Setting this to zero gives

$$\lambda = \frac{c(1+x) \pm \sqrt{c^2(1+x)^2 - 4x}}{2}$$

The eigenvalue crossing occurs at

$$c^2(1+x)^2 - 4x = 0 = x^2c^2 + x(2c^2 - 4) + c^2$$

or

$$x = \frac{4 - 2c^2 \pm \sqrt{16 - 16c^2}}{2c^2} = 2 - c^2 \pm 2\sqrt{1 - c^2}$$

The minus sign solution is the one that lies in the desired range. The other represents $a_0 + a_3$ changing sign.

Note that the general form of U here allows the eigenvalues to come in from anywhere on the unitarity circle.

Now I will lead into domain-wall fermions by playing with getting solutions to something in one more dimension.

To start consider a continuous extra dimension, and call it s. Put a step at zero by considering a Wilson hamiltonian that depends on s. For $s < 0$ use the simple positive mass D_W, but for $s > 0$ use a "negative" mass just as above for the GW case. When some eigenvalue has its real part change sign, states will get bound to the surface. This only occurs for the eigenvalues that get projected into the left half of the GW V operator. This process is also a projection, but now of these eigenvalues onto the imaginary axis. All other eigenvalues remain at large real part.

To see how the projection works, play with the toy Hamiltonian

$$H = \gamma_0 \gamma_5 \frac{d}{ds} + m\epsilon(s)\gamma_0 + i\gamma_0 \vec{p} \cdot \vec{\gamma}$$

This is an operator on the original lattice space times the fifth dimension. Here p represents the imaginary part of D_W and m the real part.

For the s derivative to cancel the mass term, one should look at solutions of the form

$$\psi = e^{-m|s|\gamma_5}\psi_0$$

For this to be normalizable (assuming $m > 0$), one wants $\gamma_5\psi_0 = +\psi_0$. Now $\gamma_0\vec{p} \cdot \vec{\gamma}$ commutes with γ_5, so it can be simultaneously diagonalized. Its eigenvalues are $\lambda = \pm i|p|$. Thus we can pick ψ_0 so that

$$\gamma_5\psi_0^\pm = +\psi_0^\pm$$
$$\gamma_0\vec{p} \cdot \vec{\gamma}\psi_0^\pm = \mp i|p|\psi_0^\pm$$

and the Hamiltonian acting on ψ gives

$$H\psi^\pm = \pm|p|\psi^\pm$$

The crucial point is that the eigenvalues of the five dimensional operator

$$\gamma_0 H = \gamma_5 \frac{d}{ds} + m\epsilon(s) + i\vec{p} \cdot \vec{\gamma}$$

are the eigenvalues of the imaginary part $i\vec{p} \cdot \vec{\gamma}$ whenever $m(s)$ changes sign, but are $O(m)$ otherwise. Thus by having $m(p, s)$ change sign for small p but not for large p, i.e. pick the initial D_W in the right "circle", then the doublers are gone.

The usual implementation takes s to a lattice with a Wilson hopping in the fifth dimension involving projections $(1\pm\gamma_5)/2$. The gauge fields are kept four dimensional. The "positive" mass side can be removed via $K \to 0$, i.e. $m \to \infty$, so the modes become surface states.

-    discussed by Shockley in 1939
-    from strongly coupled bands
-    particle and anti-particle band
-    gap closes and reopens at K_c, bottom states get stuck on walls

DW versus GW?

DW advantages:
-    just Wilson fermions, easy to implement
-    L_5 a convenient control parameter

GW advantages:
- smaller matrices
- analytically cleaner?

Do we want $K_4 = K_5$? Probably not at strong coupling.

I want to follow through the Neuberger construction with this form to compare with the above eigenvalues. Thus motivated, start with

$$D_0 = m + i\vec{p} \cdot \vec{\gamma}$$

Squaring this,
$$D_0^\dagger D_0 = m^2 + p^2$$

gives the Neuberger operator
$$D = 1 + \frac{m + i\vec{p} \cdot \vec{\gamma}}{\sqrt{p^2 + m^2}}$$

with eigenvalues
$$\lambda_\pm = 1 + \frac{m \pm ip}{\sqrt{p^2 + m^2}}$$

The real part makes this not quite the same as the domain wall approach above. For small p and $m < 0$
$$\lambda_\pm = ip/m + O(p^3)$$

To go between the GW operator and the DW one, one should look at small scales.

It would be nice to have a cleaner relation between finite L_s and the cutoff ϵ above.